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Finitary automorphism groups over commutative rings

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Abstract

We introduce a theory of finitary automorphism groups of arbitrary modules over arbitrary commutative rings, encompassing the theory of finitary linear groups (the field and vector space case). In particular, we have extended the structure theorems of U. Meierfrankenfeld et al. (J. London Math. Soc. 47 (1993) 31–40) for locally soluble finitary groups and unipotent finitary groups from the field to the commutative ring case. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Finitary linear groups over fields and more recently over division rings have received considerable attention over recent years. (See [3] for a survey of work up to about 1994.) Here we widen the scope of our investigations to cover modules over arbitrary commutative rings; here all rings have identities and all modules are unital. (Finitary groups over principal ideal domains did arise briefly towards the end of [8].)

Throughout this paper R denotes a commutative ring and M a left R-module. Set

$$F\mathrm{Aut}_R M = \{g \in \mathrm{Aut}_R M\colon M(g-1) \text{ is Noetherian as } R\text{-module}\}.$$

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This is a subgroup of $\mathrm{Aut}_R M$, to be called the finitary automorphism group of M over R; for if g and h are elements of $F\mathrm{Aut}_R M$, then

$$M(gh-1) \leqslant M(g-1)h + M(h-1)$$

and

$$M(g^{-1}-1) = Mg(g^{-1}-1) = M(g-1),$$

so both gh and g^{-1} lie in $F\mathrm{Aut}_R M$, as does the identity map on M of course. If we replace 'Noetherian' in the definition of $F\mathrm{Aut}_R M$ by 'finitely generated' we do not in general obtain a subgroup of $\mathrm{Aut}_R M$. We give an example at the end of this paper.

About locally soluble groups we can prove the following (the case for R a field can be found in the Meierfrankenfeld et al. paper [2], see especially Theorem A, Proposition 1 and Corollary 1; for some finitary skew linear analogues, see [7]).

Theorem 1. *Let M be a module over the commutative ring R.*
(a) *A locally soluble subgroup of $F\mathrm{Aut}_R M$ is hyperabelian, is locally-nilpotent by abelian by locally-finite and has a local system of soluble normal subgroups.*
(b) *Let G be any subgroup of $F\mathrm{Aut}_R M$. Then G has a unique maximal locally soluble normal subgroup, S say, S contains every normal, indeed every ascendant, locally soluble subgroup of G and S has a local system of soluble normal subgroups of G.*

The subgroup S of Theorem 1(b) we call the locally soluble radical of G and we denote it here by $1\mathrm{s}(G)$. Call an element g of $F\mathrm{Aut}_R M$ unipotent if $g-1$ is a nilpotent element of $\mathrm{End}_R M$ and call a subgroup G of $F\mathrm{Aut}_R M$ unipotent if each element of G is unipotent. Not surprisingly, unipotence plays a key role in the proof of the above theorem. The following summarizes the main properties of unipotent groups in this context. (For the special case with R a field see [2], especially Theorem B; for its skew linear analogue, see [7] Section 2.)

Theorem 2. *Let M be a module over the commutative ring R and let G be any subgroup of $F\mathrm{Aut}_R M$.*
(a) *G has a unique maximal unipotent normal subgroup, U say.*
(b) *U has a local system of nilpotent normal subgroups of G.*
(c) *U is a Fitting group and hence U is locally nilpotent and hyperabelian.*
(d) *U contains every ascendant unipotent subgroup of G.*

Of course, we call the subgroup U of Theorem 2 the unipotent radical of G and denote it by $u(G)$. Theorems 1 and 2 have the following consequences for locally nilpotent groups:

Corollary. *Let G be a locally nilpotent subgroup of $F\mathrm{Aut}_R M$, for M a module over the commutative ring R.*
(a) *Every finitely generated subgroup of G is ascendant in G; that is, G is a Gruenberg group.*
(b) *$u(G)$ is equal to the set of all unipotent elements of G.*

2. Generalities

For $g \in \mathrm{Aut}_R M$, the sequence $0 \to C_M(g) \to M \to M(g-1) \to 0$ is exact. Thus if M has no non-zero R-Noetherian images (e.g. if $R = \mathbb{Z}$ and $M = \mathbb{Q}$) or if M has no non-zero R-Noetherian submodules (e.g. if $R = M$ is a polynomial ring over a field in infinitely many variables), then $F\mathrm{Aut}_R M = \langle 1 \rangle$.

Suppose $F\mathrm{Aut}_R M \neq \langle 1 \rangle$. The set $\mathscr{A} = \{\mathrm{ann}_R(x)\colon x \in M \setminus \{0\}\}$ of annihilators has maximal members. For there exists a non-zero Noetherian R-submodule N of M. Then $S = R/\mathrm{ann}_R N$ is Noetherian (if $N = Rx_1 + \cdots + Rx_k$, then each $R/\mathrm{ann}_R(x_i) \cong Rx_i$ is Noetherian and $\mathrm{ann}_R N = \bigcap_i \mathrm{ann}_R(x_i)$; therefore S is Noetherian) and hence

$$\mathscr{B} = \{\mathrm{ann}_R(x)\colon x \in M \setminus \{0\} \text{ with } \mathrm{ann}_R N \leqslant \mathrm{ann}_R(x)\}$$

has a maximal member. Clearly a maximal member of $\mathscr{B}$ is also a maximal member of $\mathscr{A}$.

Let $\mathfrak{p}_o = \mathrm{ann}_R(x_o)$ be a maximal member of $\mathscr{A}$. Then $\mathfrak{p}_o$ is a prime ideal of R (if α, $\beta \in R$ with $\alpha\beta x_o = 0$, then either $\beta x_o = 0$ and $\beta \in \mathfrak{p}_o$, or, $\beta x_o \neq 0$, $\mathfrak{p}_o + R\alpha \leqslant \mathrm{ann}_R(\beta x_o)$ and the maximality of $\mathfrak{p}_o$ in $\mathscr{A}$ yields $\alpha \in \mathfrak{p}_o$). Set $M_1 = \mathrm{ann}_M(\mathfrak{p}_o) = \{x \in M\colon \mathfrak{p}_o x = \{0\}\}$. Then M_1 is a fully invariant (i.e. $M_1\eta \leqslant M_1$ for all η in $\mathrm{End}_R M$) R-submodule of M. Further M_1 is torsion-free as $R_o = R/\mathfrak{p}_o$-module (for if $x \in M_1 \setminus \{0\}$ and $\alpha \in R$ with $\alpha x = 0$, then $\mathfrak{p}_o + R\alpha \leqslant \mathrm{ann}_R(x)$, so $\alpha \in \mathfrak{p}_o$ by the maximality of $\mathfrak{p}_o$ again). Let F_o denote the quotient field of R_o. Then M_1 embeds into $V_o = F_o \otimes_{R_o} M_1$ and $1 \otimes F\mathrm{Aut}_{R_o} M_1 \leqslant \mathrm{FGL}({}_{F_o}V_o)$, the finitary linear group of V_o over F_o. Further $F\mathrm{Aut}_R M$ restricts on M_1 to a subgroup of $F\mathrm{Aut}_R M_1 = F\mathrm{Aut}_{R_o} M_1$. More generally we have the following:

2.1. *If G is a subgroup of $F\mathrm{Aut}_R M$ and if N is an RG-submodule of M, then $G|_N \leqslant F\mathrm{Aut}_R N$ and $G|_{M/N} \leqslant F\mathrm{Aut}_R(M/N)$.*

To see 2.1, note that if $g \in G$, then $M(g-1)$ is Noetherian as R-module and hence so too are $N(g-1) \leqslant M(g-1)$ and $(M/N)(g-1) \cong M(g-1)/(N \cap M(g-1))$.

Returning to the M and M_1 above, we can now pass to M/M_1 and, provided only that $F\mathrm{Aut}_R(M/M_1) \neq \langle 1 \rangle$, repeat the above construction of $\mathfrak{p}_o$ and M_1. A transfinite induction yields the following:

2.2. Proposition. *M has an ascending series $\{M_\sigma\}_{\sigma \leqslant \rho+1}$ of fully invariant R-submodules such that with $N_\sigma = M_{\sigma+1}/M_\sigma$ for $\sigma \leqslant \rho$ we have that $F\mathrm{Aut}_R M$ acts trivially on N_ρ and for $\sigma < \rho$ the module N_σ has prime annihilator $\mathfrak{p}_\sigma$ such that N_σ is torsion-free as $R_\sigma = R/\mathfrak{p}_\sigma$-module and for F_σ the quotient field of R_σ and $V_\sigma = F_\sigma \otimes_{R_\sigma} N_\sigma$ we have $F\mathrm{Aut}_R M|_{N_\sigma}$ embedded naturally in $\mathrm{FGL}({}_{F_\sigma}V_\sigma)$. Further we have an exact sequence*

$$1 \to \bigcap_\sigma C_G(N_\sigma) \to G \to \mathop{\times}_\sigma G|_{V_\sigma}.$$

We have only to check that in the sequence G maps into the direct product of the $G|_{V_\sigma}$ and not just into their cartesian product. If $g \in G$, then $M(g-1)$ is R-Noetherian, while the series $\{M_\sigma\}$ is ascending. Thus $M_\sigma \cap M(g-1) = M_{\sigma+1} \cap M(g-1)$ for almost all σ. For such a σ we have $M_{\sigma+1}(g-1) \leqslant M_{\sigma+1} \cap M(g-1) \leqslant M_\sigma$, so $N_\sigma(g-1) = \{0\}$ and $g|_{V_\sigma} = 1$. Thus g acts trivially on all but a finite number of the V_σ and hence G maps into the direct product of the $G|_{V_\sigma}$, as claimed.

2.3. *Let* $G = \langle g_1, g_2, \ldots, g_n \rangle$ *be a finitely generated subgroup of* $F\mathrm{Aut}_R M$. *Then*
(a) $[M, G] = \sum_{g \in G} M(g-1)$ *is R-Noetherian.*
(b) $M/C_M(G)$ *is R-Noetherian.*
(c) *There is a finitely R-generated, G-faithful RG-submodule U of M.*

In 2.3(c) we cannot in general choose U to be R-Noetherian. For example, let F be a field, $R = F[X_i\colon i = 1, 2, \ldots]$ the polynomial ring in the infinitely many variables X_i and $M = R \oplus F$, where F is made into an R-module via $X_i \mapsto 0$ for all i. Set $G = \langle g \rangle \leqslant \mathrm{Aut}_R M$, where $g\colon (f(X), \alpha) \mapsto (f(X), f(0) + \alpha)$. Then $M(g-1) = F$ and $G \leqslant F\mathrm{Aut}_R M$. Now R is a non-Noetherian domain, so $\{0\}$ is the only Noetherian R-submodule of R and F is the unique maximal Noetherian R-submodule of M. Clearly F is not G-faithful. Thus for this R and M we cannot choose a U as in 2.3(c) with U also R-Noetherian. (Of course M itself here is finitely generated, so the existence of a U as in 2.3(c) is trivial, namely take $U = M$.)

Proof. (a) $[M, G] = M(g_1 - 1) + M(g_2 - 1) + \cdots + M(g_n - 1)$, so $[M, G]$ is R-Noetherian.
(b) $M/C_M(g_i) \cong M(g_i - 1)$ is R-Noetherian, $C_M(G) = \bigcap_i C_M(g_i)$ and $M/C_M(G)$ embeds into $\bigoplus_i M(g_i - 1)$. Therefore $M/C_M(G)$ is R-Noetherian.
(c) Choose any finitely generated R-submodule U of M with $[M, G] \leqslant U$ and $M = U + C_M(G)$. □

3. Unipotence

Unipotence has been defined in Section 1.

3.1. (a) *If M is* $\mathbb{Z}$*-torsion and* $g \in F\mathrm{Aut}_R M$ *is unipotent, then g has finite order.*
(b) *If* $g \in F\mathrm{Aut}_R M$ *is unipotent of finite order, then* $\pi(g) \subseteq \pi_{\mathbb{Z}}(M)$.

Here $\pi(g)$ denotes the set of prime divisors of $|g|$ and $\pi_{\mathbb{Z}}(M)$ the prime spectrum of M as $\mathbb{Z}$-module.

Proof. (a) By 2.3(c) we may assume that M is finitely R-generated and hence by [5, 13.4] that R is Noetherian. Then M has finite exponent as $\mathbb{Z}$-module. Now $\langle g \rangle$ stabilizes a series of R-submodules in M of finite length ([5, 13.6] but obvious anyway). By stability group theory g has finite order.

(b) Let $C = C_M(g)$. Then $\pi_{\mathbb{Z}}(M/C) \subseteq \pi_{\mathbb{Z}}(M)$; for if $x \in M$ and $m \in \mathbb{Z}$ with $mx \in C$ and with m prime to $\pi_{\mathbb{Z}}(M)$, then $m(x(g-1)) = 0$, so $x(g-1) = 0$ and $x \in C$. We prove

3.1(b) by induction on the least n with $M(g-1)^n = \{0\}$. By induction $\pi(g|_{M/C}) \subseteq \pi_{\mathbb{Z}}(M/C) \subseteq \pi_{\mathbb{Z}}(M)$. If $e = |g|_{M/C}|$ and $h = g^e$, then h stabilizes the series $\{0\} \leqslant C \leqslant M$, so $\langle h \rangle$ embeds into $\mathrm{Hom}_R(M/C, C)$. Clearly then, $\pi(h) \subseteq \pi_{\mathbb{Z}}(C) \subseteq \pi_{\mathbb{Z}}(M)$. The result follows. □

Usually in finitary situations stability groups (of series of submodules) are unipotent, with the converse holding in very many situations (e.g. if R is a field or e.g. if R is a division ring of characteristic zero), and in practice stability groups are the 'right' analogue of unipotent linear groups of finite degree. Over commutative rings, however, this breaks down. For a simple example consider the following: Let $R = \mathbb{Z} = M$ and let $G = \langle -1 \rangle$. Then G stabilizes the series

$$M \supseteq 2M \supseteq 2^2M \supseteq \cdots \supseteq 2^iM \supseteq \cdots \supseteq \{0\}$$

but is not unipotent. This is not an ascending series, of course, but with infinite-dimensional finitary linear groups one certainly does not want to restrict oneself to ascending series, so one should not want to here either.

We use the standard $(\Lambda_\alpha, V_\alpha)_{\alpha \in I}$ notation of P. Hall for series, where the Λ_α/V_α are the factors of the series, see [4, Vol. 1, p. 9]. Call a series $(\Lambda_\alpha, V_\alpha)_I$ of R-submodules of M (with $\bigcup_\alpha \Lambda_\alpha = M$ and $\bigcap_\alpha V_\alpha = \{0\}$) *finitary for the subgroup* G of $F\mathrm{Aut}_R M$ (or *G-finitary* for short) if the set $\{\Lambda_\alpha \cap M(g-1)\colon \alpha \in I\}$ is finite (equivalently if the set $\{V_\alpha \cap M(g-1)\colon \alpha \in I\}$ is finite) for every g in G. If the series is finitary for $F\mathrm{Aut}_R M$ itself, we simply say the series is *finitary*. Since $M(g-1)$ is R-Noetherian for every $g \in F\mathrm{Aut}_R M$, so every ascending R-series of M is finitary. If R is a field then each $M(g-1)$ is also Artinian and then every R-series of M is finitary.

3.2. *Let G be a subgroup of $F\mathrm{Aut}_R M$.*
(a) *If G is unipotent, then G acts unipotently on every RG-section of M.*
(b) *G is unipotent if and only if G stabilizes a finitary series of R-submodules of M.*

Proof. (a) This is immediate from the definitions.

(b) Suppose G is unipotent. Apply 2.2. Then for each $\sigma < \rho$ the group $G|_{V_\sigma} \leqslant FGL(V_\sigma)$ is unipotent and hence stabilizes an F_σ-series in V_σ (see [2, Theorem B or 7, 2.1d]). Hence G stabilizes an R-series in N_σ, and this is trivially true for $\sigma = \rho$. (Since $\dim_{F_\sigma}[V_\sigma, g]$ is finite for each g in G, this also gives a finitary such series of N_σ.) These series yield an R-series of M stabilized by G.

For $\sigma < \rho$ and $g \in F\mathrm{Aut}_R M$, the R_σ-module $N_\sigma(g) = (M(g-1) \cap M_{\sigma+1} + M_\sigma)/M_\sigma$ is finitely generated, so $\dim_{F_\sigma} F_\sigma \otimes_{R_\sigma} N_\sigma(g)$ is finite. Thus any R-series, for example the one constructed above stabilized by G, which is constructed from F_σ-series of V_σ for each σ as above, is finitary; this depends, of course, on the fact that $\{M_\sigma\}_{\sigma \leqslant \rho}$ being an ascending series, almost all the $N_\sigma(g)$ for a given g are trivial.

Now suppose that G stabilizes a finitary R-series in M. Then any g in G stabilizes a finite R-series in $M(g-1)$. Hence $M(g-1)(g-1)^n = \{0\}$ for some n and therefore g, and consequently G, are unipotent. □

3.3. *Let N be a unipotent normal subgroup of the subgroup G of $F\mathrm{Aut}_R M$. Then N stabilizes a finitary series of RG-submodules of M.*

Proof. Repeat the proof of 3.2(b), using [7, 2.2a] in place of [2, Theorem B] (or [7, 2.1(d)]). $\square$

3.4. *Let G be a subgroup of $F\mathrm{Aut}_R M$ and let $\{L_i: i \in I\}$ be the factors of some RG-series S of M. Set $L = \bigoplus_{i\in I} L_i$. If S is finitary for G, then G acts finitarily on L. In particular, G acts finitarily on L whenever S is finitary.*

Proof. For any g in G, clearly $L(g-1) = \bigoplus_I L_i(g-1)$. The latter is R-Noetherian if and only if g acts trivially on all but a finite number of the L_i. If S is finitary for $\langle g \rangle$, the latter holds. $\square$

Remark. The converse of 3.4 does not hold and the usual example is a counterexample; viz. let $R = \mathbb{Z} = M$ and let $G = \langle -1 \rangle$. Then G acts trivially on all the factors of the series

$$M \supseteq 2M \supseteq 2^2 M \supseteq \cdots \supseteq 2^i M \supseteq \cdots \supseteq \{0\}$$

and yet this series is not finitary; clearly for $g = -1$ it intersects $M(g-1) = 2M$ in an infinite series. Alternatively, a converse to 3.4, with 3.2, would imply that every stability group is unipotent, a conclusion we have already seen is false.

3.5. *Let G be a subgroup of $F\mathrm{Aut}_R M$. Then the following hold*:
(a) *G has a unique maximal unipotent normal subgroup, $u(G)$ say.*
(b) *$u(G)$ contains every unipotent normal subgroup of G.*
(c) *$G/u(G)$ is isomorphic to some finitary group G_1 over R with $u(G_1) = \langle 1 \rangle$.*
(d) *There is an exact sequence $1 \to u(G) \to G \to H \to 1$, where H is a subdirect product of irreducible finitary linear groups.*

We call $u(G)$ the unipotent radical of G.

Proof. (a) & (b). We apply 2.2. With the notation there, choose for each $\sigma < \rho$ an $F_\sigma G$-composition series for V_σ. These give RG-series in each N_σ and fit together to give a finitary RG-series S for M (add the term M if necessary), cf. the proof of 3.2(b). Let $u(G)$ be the stabilizer of the series S. By 3.2(b) the subgroup $u(G)$ is unipotent. Since S consists of G-submodules of M, so $u(G)$ is normal in G.

Let U be any unipotent normal subgroup of G. Trivially U acts trivially on M/M_ρ. Let W be a factor of S between M_σ and $M_{\sigma+1}$ for some $\sigma < \rho$. Then $F_\sigma \otimes_{R_\sigma} W$ is an $F_\sigma G$-irreducible module upon which U acts unipotently. By 3.3 (or directly from [7, 2.2a]) the subgroup U acts trivially on W. Thus $U \leqslant u(G)$. Parts (a) and (b) of 3.5 now follow.

(c) Let L be the sum of the factors in the series S in the above construction of $u(G)$. By 3.4 the group G acts finitarily on the R-module L and by definition $u(G)$ acts trivially on L. If U is a normal subgroup of G acting unipotently on L, then

U acts trivially on each factor of S (M/M_ρ by 2.2 and the other factors by 3.3), so $U \leqslant u(G)$ and hence $G/u(G)$ acts faithfully on L with trivial unipotent radical.

(d) By 2.2 we have $1 \to \bigcap_\sigma C_G(N_\sigma) \to G \to \times_\sigma FGL(V_\sigma)$ exact. Clearly $\bigcap_\sigma C_G(N_\sigma)$ lies in $u(G)$ and $u(G)$ is the intersection of the inverse images in G of the unipotent radicals of the images of G in the $FGL(V_\sigma)$. The result follows from the finitary linear case ([7, 2.2b]). □

3.6. *Let X be a finite subset of the subgroup G of $F\mathrm{Aut}_R M$ and set $U = u(\langle X^G \rangle)$. Then U is nilpotent.*

Now $Y = [M, \langle X \rangle\}$ is R-Noetherian (2.3), so in the notation of 2.2 above, almost all of the $(M_{\sigma+1} \cap Y)/(M_\sigma \cap Y)$ are zero. Then

$$n = \sum_{\sigma < \rho} \dim_{F_\sigma} F_\sigma \otimes_{R_\sigma} ((M_{\sigma+1} \cap Y) + M_\sigma)/M_\sigma$$

is finite. We prove that U is nilpotent of class at most $2n$ in general and at most n if $X \subseteq u(G)$.

Proof. Choose an $F_\sigma G$-composition series of V_σ for each $\sigma < \rho$, pull back to N_σ and then run them together to obtain a refinement of the series $\{M_\sigma\}_{\sigma \leqslant \rho+1}$ of 2.2. By definition of n, the submodule Y must avoid all but n of the factors of this series. Thus for some $m \leqslant n$ we have an RG-series

$$\{0\} = Q_0 \leqslant P_1 < Q_1 \leqslant \cdots \leqslant P_m < Q_m \leqslant P_{m+1} = M \qquad (*)$$

of M such that Y avoids each factor P_{i+1}/Q_i and for each i there is a prime ideal $\mathfrak{p}_i$ of R such that if F_i denotes the quotient field of $R_i = R/\mathfrak{p}_i$, then $V_i = F_i \otimes_{R_i} (Q_i/P_i)$ is $F_i G$-irreducible.

Now $[P_{i+1}, \langle X \rangle] \leqslant Q_i$ for each i and the series $(*)$ is a G-series. Hence $[P_{i+1}, U] \leqslant [P_{i+1}, \langle X^G \rangle] \leqslant Q_i$ for each i. By 3.5(a) (uniqueness) the subgroup U is normal in G. Hence 3.3 implies that $[Q_i, U] \leqslant P_i$ for each i. Thus U stabilizes the series $(*)$ of length $2m+1$ and hence U is nilpotent of class at most $2m \leqslant 2n$.

If $X \subseteq u(G)$, then $[Q_i, \langle X \rangle] \leqslant P_i \cap Y \leqslant Q_{i-1}$, so $U = \langle X^G \rangle$ stabilizes the series

$$\{0\} < Q_1 < Q_2 < \cdots < Q_m \leqslant M$$

of length $m+1$ and hence U is nilpotent of class at most $m \leqslant n$. □

3.7. (a) *If $G \leqslant F\mathrm{Aut}_R M$, then $u(G)$ lies in the Fitting subgroup of G.*

(b) *Let G be unipotent subgroup of $F\mathrm{Aut}_R M$. Then G is a Fitting group and hence is locally nilpotent and hyperabelian.*

Proof. Part (a) follows from 3.6 and Part (b) follows from Part (a). □

3.8. *If H is an ascendant subgroup of the subgroup G of $F\mathrm{Aut}_R M$, then $u(H)$ is an ascendant subgroup of $u(G)$.*

Proof. Clearly $u(H)$ is ascendant in G, so we need only prove that $u(H)$ is a subgroup of $u(G)$. Suppose

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_\alpha \lhd \cdots \lhd H_\gamma = G$$

is an ascending series of G. We induct on γ. Suppose $u(H_\alpha) \leqslant u(H_\beta)$ whenever $\alpha \leqslant \beta < \gamma$. If $\gamma - 1 = \beta$ exists, then $u(H_\beta)$ is normal is H_γ by 3.5(a) and hence $u(H_\beta) \leqslant u(H_\gamma)$ by 3.5(b). Suppose γ is a limit ordinal. Then $U = \bigcup_{\alpha<\rho} u(H_\alpha)$ is a unipotent normal subgroup of H_γ, so $U \leqslant u(H_\gamma)$ by 3.5(b) again. It now follows that $u(H_\alpha) \leqslant u(H_\beta)$ for all $\alpha \leqslant \beta \leqslant \gamma$ and hence $u(H) \leqslant u(G)$ as required. □

Note that Theorem 2 follows from 3.5(a), 3.6, 3.7(b) and 3.8.

4. Local solublity

4.1. *Let G be any subgroup of $F\mathrm{Aut}_R M$.*

(a) *If $\mathfrak{X}$ is any class of groups satisfying $Q\mathfrak{X} = \mathfrak{X}$ and $\mathfrak{X} \subseteq \mathrm{L}(\mathfrak{G} \cap \mathfrak{X})$ and $\mathfrak{X} \cap \mathfrak{F} \subseteq \mathfrak{S}$ and if $G \in \mathfrak{X}$, then G is locally soluble.*

(b) *If $G \in \langle \mathrm{P}, \mathrm{L}\rangle\mathfrak{A}$, then G is locally soluble. In particular, if G is an extension of a locally soluble group by a locally soluble group, then G is locally soluble.*

(c) *G has a unique maximal locally soluble normal subgroup*, $1\mathrm{s}(G)$ say.

(d) *If H is an ascendant subgroup of G, then $1\mathrm{s}(H)$ is an ascendant subgroup of* $1\mathrm{s}(G)$.

Proof. (a) Apply [5, 13.13] to the finitely generated $\mathfrak{X}$-subgroups of G via 2.3(c).

(b) The class $\langle \mathrm{P}, \mathrm{L}\rangle\mathfrak{A}$ is an example of a class $\mathfrak{X}$ as in Part (a) and it contains every extension of a locally soluble group by a locally soluble group.

(c) This follows from (b).

(d) A straightforward transfinite induction shows that $\langle 1\mathrm{s}(H)^G\rangle \in \langle \mathrm{P}, \mathrm{L}\rangle\mathfrak{A}$. Thus (d) follows from (b) and (c). □

4.2. *Let $S = 1\mathrm{s}(G)$ be the locally soluble radical of the subgroup G of $F\mathrm{Aut}_R M$. If X is any finite subset of S, then $\langle X^G\rangle$ is soluble. Thus S has a local system of soluble normal subgroups of G and an ascending series of normal subgroups of G (including $\langle 1\rangle$ and S) with abelian factors.*

Proof. Let $Y = \langle X^G\rangle$. By 3.6 the subgroup $u(Y)$ is nilpotent. We apply 2.2. By [2, Proposition 2], for each $\sigma < \rho$, the image of Y in $FGL(V_\sigma)$ is soluble. Moreover, since $[M, \langle X\rangle]$ is R-Noetherian (by 2.3(a)) and the series $\{M_\sigma\}$ is ascending, for almost all σ the set X and hence the subgroup Y act trivially on N_σ and consequently on V_σ. Further,

$$Y \cap \bigcap_\sigma C_G(N_\sigma) \leqslant Y \cap u(G) \leqslant u(Y).$$

Thus $Y/u(Y)$ is soluble. Hence so too is Y. The remainder of 4.2 follows immediately. □

4.3. *Let G be a locally soluble subgroup of $F\mathrm{Aut}_R M$.*
(a) *G is unipotent by abelian by locally-finite, so in particular G is locally-nilpotent by abelian by locally-finite.*
(b) *G has a local system of soluble normal subgroups.*
(c) *G is hyperabelian.*

Note that Theorem 1 follows from 4.3, 4.1(c), 4.1(d) and 4.2.

Proof. (a) This follows from 2.2 and the finitary linear case [2, Theorem A(vi)].
(b) & (c) These follow from 4.2. □

4.4. *Let G be a subgroup of $F\mathrm{Aut}_R M$. Then the four canonical sets of Engel elements of G (left, right, bounded left and bounded right) are subgroups of G. Specifically, and in the notation of* [5] (*see alternatively* [4], *especially Vol.* 2, *Chapter* 7), *we have that the set $L(G)$ is equal to both the Hirsch–Plotkin and the Gruenberg radicals of G, the set $\bar{L}(G)$ is equal to the Baer radical of G, the set $R(G)$ is equal to $\rho(G)$ and the set $\bar{R}(G)$ is equal to $\bar{\rho}(G)$.*

Proof. Set $H = \langle L(G), R(G)\rangle$. If M is finitely R-generated, the result holds locally by [5, 13.4 & 13.18]. Hence by 2.3(c) the subgroup H is locally nilpotent. By 4.2 the group G satisfies the hypotheses of [6] 4.1. The result follows. □

4.5. *Let G be a locally nilpotent subgroup of $F\mathrm{Aut}_R M$.*
(a) *Every finitely generated subgroup of G is ascendant in G.*
(b) *The subgroup $u(G)$ is the set of unipotent elements of G.*

The corollary of Section 1 follows from 4.5.

Proof. (a) It is immediate from 4.4 that G is its own Gruenberg radical, from which Part (a) follows.

(b) Let X be a finite set of unipotent elements of G. Then $H = \langle X\rangle$ is nilpotent, so if $x \in X$, then $\langle x\rangle$ is subnormal in H. Thus by 3.8 we have $\langle x\rangle = u(\langle x\rangle) \leqslant u(H)$ and hence $H = u(H)$ is unipotent. It follows that $\langle x \in G : x \text{ unipotent}\rangle$ is unipotent. Part (b) follows. □

4.6. Remark. Let G be a simple periodic subgroup of $F\mathrm{Aut}_R M$. Then by 3.5(d) and 3.7 the group G acts faithfully on at least one of the V_σ of 2.2 and hence G is isomorphic to a simple periodic finitary linear group. The latter have been completely classified by Hall, see [1] for a description.

5. An example

Let M be a module over the commutative ring R and set

$$X = \{g \in \mathrm{Aut}_R M\colon M(g-1) \text{ is finitely generated as } R\text{-module}\}.$$

If M is finitely generated, then $X = \mathrm{Aut}_R M$. If R is Noetherian, then $X = F\mathrm{Aut}_R M$. Thus in these cases X is a subgroup of $\mathrm{Aut}_R M$. However in general X need not be a subgroup.

Let F be any field, let R denote the polynomial ring $F[X_i\colon i=1,2,\dots]$ in the infinitely many variables X_i and let M be the free R-module on the basis $\{e_i\colon i=0,1,2,\dots\}$. Let $g,h \in \mathrm{Aut}_R M$ be defined by

$$e_0 g = e_0, \qquad e_1 g = e_0 + e_1 \quad \text{and} \quad e_i g = X_i e_0 + e_i \quad \text{for } i \geqslant 2$$

and

$$e_0 h = e_0, \qquad e_1 h = (X_1 - 1)e_0 + e_1 \quad \text{and} \quad e_i h = e_i \quad \text{for } i \geqslant 2.$$

Then $e_0 gh = e_0$ and $e_i gh = X_i e_0 + e_i$ for $i \geqslant 1$. Hence $M(g-1) = Re_0$ and $M(h-1) = R(X_1 - 1)e_0$ are finitely generated, while $M(gh-1) = (\sum_{i \geqslant 1} RX_i)e_0$ is not. Thus

$$\{x \in \mathrm{Aut}_R M\colon M(x-1) \text{ is finitely generated as } R\text{-module}\}$$

is not a subgroup of $\mathrm{Aut}_R M$ in this case.

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